Structural Equation Modeling: A Multidisciplinary Journal

ISSN: 1070-5511 (Print) 1532-8007 (Online) Journal homepage: https://www.tandfonline.com/loi/hsem20

Factor Score Regression in Connected Measurement Models Containing Cross-Loadings

Timothy Hayes & Satoshi Usami

To cite this article: Timothy Hayes & Satoshi Usami (2020): Factor Score Regression in Connected Measurement Models Containing Cross-Loadings, Structural Equation Modeling: A Multidisciplinary Journal, DOI: 10.1080/10705511.2020.1729160

To link to this article: https://doi.org/10.1080/10705511.2020.1729160

STRUCTURAL EQUATION MODELING: A MULTIDISCIPLINARY JOURNAL
https://doi.org/10.1080/10705511.2020.1729160

Factor Score Regression in Connected Measurement Models Containing Cross-Loadings

Timothy Hayes[a] and Satoshi Usami[b]

[a]Florida International University; [b]University of Tokyo

ABSTRACT

Factor Score Regression (FSR) methods have received increased interest in the quantitative literature, with Croon's *bias-correcting* method gaining particular traction. By fixing measurement parameters in place in an initial step, FSR methods aim to stymie the proliferation of bias in larger structural models that may contain misspecification. Although Croon's approach was originally derived for factor models exhibiting simple structure and conditionally independent unique factors, Hayes and Usami recently extended this method to connected measurement models featuring correlated uniquenesses. In this article, we demonstrate that their formulas also correct bias in models that feature cross-loadings. We begin by discussing bias in SEMs that incorrectly impose simple structure. We then describe Croon's approach in connected measurement models featuring cross-loadings and compare its performance to two other state-of-the-art FSR approaches both analytically and via a simulated demonstration.

KEYWORDS
Factor score regression; structural equation modeling; cross-loadings; measurement

By incorporating both measurement and structural components, Structural Equation Models (SEMs) allow researchers to obtain error-free estimates of the true score, directed relationships among latent constructs of theoretical interest. The power of the SEM framework comes at a cost, however, since unbiased simultaneous estimation hinges on the correct specification of every aspect of the model. Although the risk of model misspecification can never be eliminated, the extent of its damage can be reduced by switching from a simultaneous to a multi-stage estimation approach such as *Factor Score Regression* (*FSR*; Croon, 2002; Devlieger, Mayer, & Rosseel, 2016; Devlieger & Rosseel, 2017; Hoshino & Bentler, 2013; Lu, Kwan, Thomas, & Cedzynski, 2011; Skrondal & Laake, 2001). In FSR, factor scores are extracted from each factor model in an initial step and then used as input data in a subsequent regression (or path analysis; see Devlieger & Rosseel, 2017). By specifying and extracting factor scores from each factor model separately, FSR freezes the measurement model parameters at their initial estimates, effectively blocking any parameter drift that would result from misspecification elsewhere in the model.

Recent presentations of FSR methods (e.g., Devlieger et al., 2016; Devlieger & Rosseel, 2017; Devlieger, Talloen, & Rosseel, 2019; Lu et al., 2011) have focused on models in which (a) all factors exhibit *simple structure* (Thurstone, 1935, 1947), with each indicator loading on only one factor, and (b) all unique factors exhibit conditional independence. When an SEM features a *connected measurement model* in which the individual factor models are joined by either cross-loading indicators or correlated unique factors, however, factor score extraction in step 1 of an FSR must proceed at the level of the connected measurement model rather than the individual factor models (cf. Hayes & Usami, 2020; Skrondal & Laake, 2001).

In this article, we first utilize two template models to illustrate the manner in which structural regression parameters may become biased in misspecified SEMs that ignore cross-loadings and impose simple structure. We then compare and contrast the performance of three cutting-edge FSR methods (Croon, 2002; Hayes & Usami, 2020; Hoshino & Bentler, 2013; Skrondal & Laake, 2001) in the context of connected measurement models and present a focused demonstration of the methods' performance.

Bias in models assuming simple structure

Figure 1a,b presents two prototypical structural regression models involving a connected measurement model consisting of two latent factors with four primary indicators each, some of which potentially exhibit cross-loadings (denoted by the gray solid and dotted lines). Such a scenario could easily arise, for example, if an eight-item questionnaire consists of two 4-item subscales, with certain items displaying nonzero cross-loadings. Once the measurement model is estimated, a typical researcher might be interested in ascertaining whether the latent factors (e.g., subscales) are either predicted by or predictive of other construct(s) of interest. In line with this reasoning, Figure 1a presents a model (Model A) in which a third variable, W, is positioned as a *common cause* of latent variables η_1 and η_2. Figure 1b presents an alternative model (Model B) in which third variable W is positioned as a *common outcome* of both factors.

In the following sections, we describe the bias that would result in Models A and B if an analyst mistakenly imposed a simple structure model when, in fact, some or all of the gray

CONTACT Timothy Hayes timothybhayes@gmail.com Psychology Department, Florida International University, 11200 SW 8 Street, DM 381B, Miami, FL 33199.
Supplemental data for this article can be accessed here.

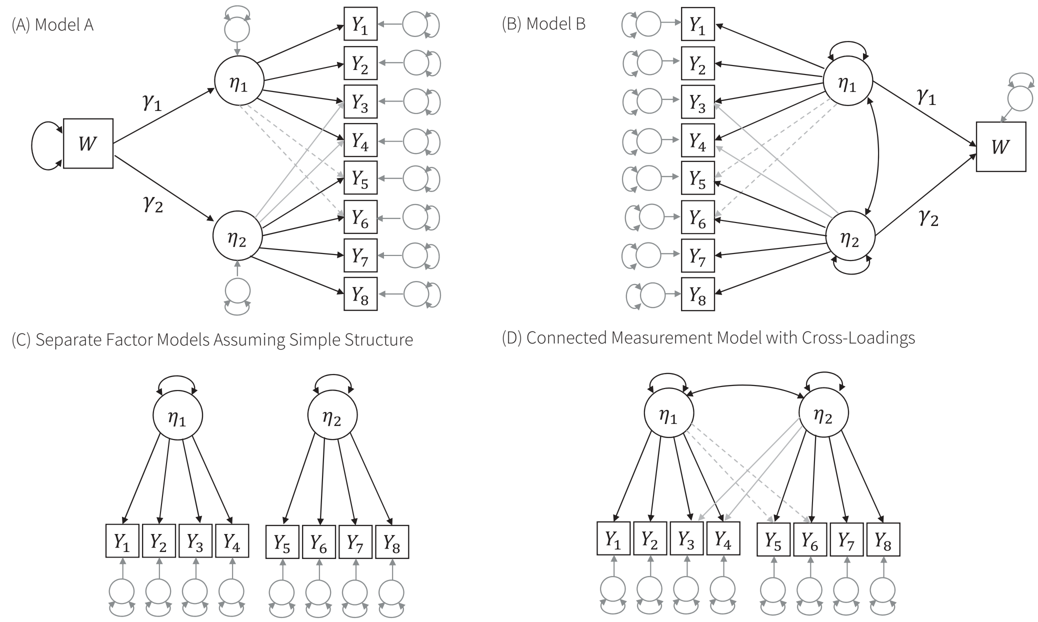

Figure 1. (a) Model A: Connected measurement model with common predictor, W. (b) Model B: Connected measurement model with common outcome, W. (c) Individual factor models assuming simple structure. (d) Connected measurement model featuring cross-loadings. Gray lines (both solid and dotted) indicate potentially nonzero cross-loadings in the hypothetical population model used in our running example and simulation study. Solid lines are assumed nonzero in the population in all models discussed. Dashed lines are assumed nonzero in the population in our discussion of measurement models in which both primary factors are measured by indicators that cross-load.

cross-loadings in Figure 1 are truly nonzero.[1] To simplify the algebraic exposition, we first describe the propagation of bias in the models of Figure 1, in which a manifest third variable predicts a connected measurement model comprised two latent factors. Later, we demonstrate these same principles in the context of the richer measurement model of Figure 2, in which third variable W is a latent (rather than manifest) variable measured by four indicators.

Bias in the estimated factor correlation

As stated by Asparouhov and Muthén (2009), "when nonzero cross-loadings are specified as zero, the correlation between factor indicators representing different factors is forced to go through their main factors only, usually leading to overestimated factor correlations and subsequent distorted structural relations" (p. 398). Additionally, we note the magnification of the estimated factor correlation, $r_{\eta_1\eta_2}$, will exert pressure to

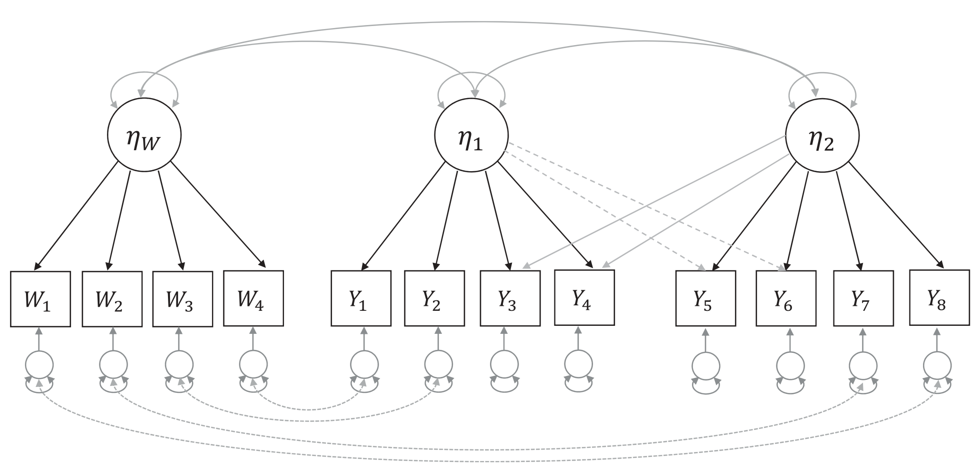

Figure 2. Connected measurement model used in the simulated demonstrations, with all three factors latent. Dashed lines represent parameters that are nonzero only in certain simulated conditions.

[1]Throughout these sections as well as the online supplemental appendices, we make the simplifying assumptions that the latent factor correlation and structural regression coefficients involving third variable *W* are all positive and that all factor loadings, including cross-loadings, are positive (as would occur, e.g., when an indicator of a latent "optimism" factor exhibits a positive cross-loading with a latent factor representing "life satisfaction"). In another plausible case, the factor correlation could be negative, in which case the cross-loadings would likely be negative (e.g., a positively-loading indicator of "optimism" cross-loads negatively on a latent "depression" factor). In such a case, the direction of parameter bias in each estimate may differ from what is described here. We also note that the derivations that follow assume all latent factors are standardized with zero means and unit variances, but analogous results should hold in measurement models that employ Unit Loading Identification (ULI) and unstandardized constructs.

shrink the factor loadings corresponding to indicators that do not cross-load. Take, for example, indicators Y_1 and Y_7 in the models of Figure 1. The model implied covariance between these indicators, derived, e.g.,, using covariance algebra (Kenny, 1979) or path tracing rules (Wright, 1934), is $\sigma_{Y_1Y_7} = \lambda_{Y_1} r_{\eta_1\eta_2} \lambda_{Y_7}$, where $\lambda_{Y_1}$and $\lambda_{Y_7}$are factor loadings for indicators Y_1 and Y_7 . It is clear from this equation that if the middle term, $r_{\eta_1\eta_2}$, is magnified, one or both of the other terms will necessarily need to be attenuated if the product is to accurately reproduce the sample covariance between Y_1 and Y_7.

Bias in the structural regression coefficients and disturbance covariance in common cause Model A

What would happen if an analyst imposed a simple structure model for η_1 and η_2 when, in fact, some of the cross-loadings (indicated by gray one-headed arrows) in Figure 1a were actually nonzero in the population? Take as an example the indicators of the first factor. If simple structure is imposed in common cause Model A of Figure 1a, the model-implied covariance between any indicator, Y_i, of η_1 and third variable W will be $\sigma_{Y_iW} = \lambda_{Y_i}\gamma_1$, yielding an algebraic estimator of the structural coefficient as $\gamma_1 = \sigma_{Y_iW}/\lambda_{Y_i}$. As noted above, the factor loadings for indicators like Y_1 that do not cross-load are generally attenuated in models with omitted cross-loadings, leading to magnified estimates of γ_1 in the previous equation due to division by a shrunken denominator.

Indicators such as Y_3 or Y_4 that actually cross-load in the population also exert pressure to magnify the estimate of γ_1, but for different reasons. For indicators like Y_3, the true model-implied covariance with third variable W in the population is a function of both the indicator's relationship to η_1, through its primary loading, and its relationship to η_2, through its cross-loading (for precise formulas and algebra, see the online supplemental appendices). Mirroring the logic described by Asparouhov and Muthén (2009), the imposition of a simple structure model forces the relationship between Y_3 and third variable W to be channeled entirely through primary factor η_1, resulting in pressure to magnify the estimate of γ_1 during estimation. Because in fully standardized models a bivariate regression coefficient is equivalent to the bivariate correlation between the predictor and outcome variables, saying that the estimates of γ_1 and γ_2 are tantamount to saying that the estimates of r_{η_1W} and r_{η_2W} are magnified in simple structure models omitting truly nonzero cross-loadings.

Finally, we note that in addition to the inflated structural regression coefficients, the estimate of the disturbance covariance between endogenous η_1 and η_2 in Model A should also be magnified if the cross-loadings are left out of the model. If, for example, the true disturbance covariance is 0 in the correctly specified model of Figure 1a, implying conditional independence of the factors after their prediction by W, it may be estimated as nonzero in a misspecified model imposing simple structure. If the degree of magnification is large enough, a model that *correctly* specifies zero disturbance covariance between η_1 and η_2 may exhibit significantly worse fit (e.g., via a chi-square difference test) than one that *incorrectly* estimates this disturbance covariance as a free parameter. In this way, structural respecifications may seem to improve the model when, in reality, they merely serve to mask the true source of the misfit.

Bias in the structural regression coefficients in common outcome Model B

What would happen if simple structure was mistakenly imposed in Model B, in which third variable W is depicted as a common outcome of η_1 and η_2? If η_1, η_2, and W are all standardized and if, further, an estimate of the covariance matrix of these variables is used as input in a path model (as would be the case in an FSR analysis), γ_1 and γ_2 will be estimated via the well-known formulas for standardized partial regression coefficients with two predictors as:

$$\gamma_1 = \frac{r_{\eta_1W} - r_{\eta_2W} r_{\eta_1\eta_2}}{1 - r^2_{\eta_1\eta_2}} \qquad (1)$$

and

$$\gamma_2 = \frac{r_{\eta_2W} - r_{\eta_1W} r_{\eta_1\eta_2}}{1 - r^2_{\eta_1\eta_2}} \qquad (2)$$

respectively. Although we leave the algebraic details for the online supplemental appendices, we note that the covariance expectations (path tracings) that comprise the identities for γ_1 and γ_2 in a simultaneous SEM model actually reduce to Equations (1) and (2) when the model is correctly specified.

As a first scenario, imagine that the gray lines, but not the gray-dotted lines, in Figure 1b are nonzero in the population – that is, that indicators Y_3 and Y_4 cross-load on η_2 whereas none of the indicators Y_5–Y_8 cross-load on η_1 – but an analyst mistakenly fits a model assuming simple structure of both factors. Because the factor model for η_2 is correctly specified, the estimate of r_{η_2W} should be fairly accurate but both of the other correlations (r_{η_1W} and $r_{\eta_1\eta_2}$) will be inflated. Because r_{η_1W} will be magnified in the numerator of Equation (1) and because the term $r_{\eta_2W} r_{\eta_1\eta_2}$ will not be comparably magnified when r_{η_2W} is not also magnified, the numerator of Equation (1) will tend to be inflated. Furthermore, because $r^2_{\eta_1\eta_2}$ will be larger than its true value, the term $1 - r^2_{\eta_1\eta_2}$ will produce an attenuated denominator, increasing the tendency toward inflation of γ_1. By contrast, because the numerator of Equation (2) subtracts the product of two inflated coefficients ($r_{\eta_1W} r_{\eta_1\eta_2}$) from a non-inflated one (r_{η_2W}), γ_2 should exhibit the reverse tendency: the numerator of this coefficient should be attenuated when some indicators of η_1, but no indicators of η_2, cross-load in Model B. If the degree of attenuation in the numerator exceeds that in the denominator, then γ_2 will be attenuated overall.

As a second scenario, imagine that both the gray solid lines and the gray-dotted lines are nonzero in the population model of Figure 1b – that is, indicators Y_3 and Y_4 cross-load on η_2 and indicators Y_5 and Y_6 cross-load on η_1. In this case, the estimates of all three correlations (r_{η_1W}, r_{η_2W}, and $r_{\eta_1\eta_2}$) will be magnified. All else being equal, the degree of magnification of the products $r_{\eta_2W} r_{\eta_1\eta_2}$ and $r_{\eta_1W} r_{\eta_1\eta_2}$ on the right-hand side of the subtraction in the numerators of (1) and (2), respectively, will likely be greater in absolute magnitude than the single correlations on the left-hand side of the subtraction, with the result that the numerators of both γ_1 and γ_2 will likely be attenuated in Model B when the indicators of both factors cross-load. If the degree of attenuation in each numerator exceeds that in the denominator, then γ_1 and γ_2 will be attenuated overall.

FSR approaches for connected measurement models

As the preceding discussion implies, the structural parameters in an SEM analysis quickly grow biased when researchers incorrectly impose simple structure on measurement models that do not adhere to it. Luckily, a variety of methods have been developed to help researchers diagnose and estimate nonzero cross-loadings, ranging from classical exploratory factor analysis (EFA; cf. Mulaik, 2009; Thurstone, 1935, 1947) to more modern approaches such as exploratory (Asparouhov & Muthén, 2009; Marsh, Morin, Parker, & Kaur, 2014; Morin, Marsh, & Nagengast, 2013), Bayesian (Asparouhov, Muthén, & Morin, 2015; Muthén & Asparouhov, 2012), or even regularized (Jacobucci, Grimm, & McArdle, 2016; Scharf & Nestler, 2019) SEM. Because the exposition of these methods is far beyond the focused scope of the present article, we simply note that once researchers have identified nonzero cross-loadings in a connected measurement model using whichever of these methods they prefer, they may wish to incorporate these connected measurement structures into FSR analyses. In this section, we review simple structure and connected measurement applications in three state of the art FSR methods: (1) the *bias-avoiding* approach of Skrondal and Laake (2001), (2) Croon's (2002) *bias-correcting* approach, and (3) Hoshino and Bentler (2013) method.

Skrondal and Laake's (2001) bias-avoiding approach

Because factor scores are imperfect, indeterminate estimators of their true population quantities (Steiger & Schönemann, 1978), coefficients from regressions or path analyses conducted directly on extracted factor scores are prone to bias, as demonstrated repeatedly in simulation studies (Devlieger et al., 2016; Devlieger & Rosseel, 2017; Hayes & Usami, 2020; Lu et al., 2011). Skrondal and Laake (2001) noted that this bias might be avoided entirely if analysts strategically extract factor scores using Bartlett estimation for all outcomes in a given analysis (Bartlett, 1937) and regression estimation for all predictors (Thurstone, 1935). As described in their original paper, regression coefficients remain unbiased whether factor scores are extracted on a factor-by-factor basis under simple structure (termed "factorwise FSR" in their paper[2]) or extracted from connected measurement "blocks" of all predictors and all outcomes, respectively (termed "blockwise factor score extraction" in their paper).

Thus, since its inception, the Skrondal-Laake (hereafter *SL*) FSR method has been capable of handling connected measurement models that may include cross-loadings or correlated uniquenesses. An important caveat, however, is that for the SL method to work, measurement models may *only* be connected within blocks but not across blocks. That is, if there are cross-loadings or correlated uniquenesses among the predictors (as in the model of Figure 1b) or among the outcomes (as in Figure 1a) but no cross-loadings or unique factor covariances across the predictor and outcome blocks, the SL method of estimating the structural regression coefficients will remain unbiased.

If there are cross-loadings between any predictors and outcomes, however, the method breaks down. For example, imagine that a researcher wished to regress η_2 on η_1 given any of the models displayed in Figures 1 and 2. In such a scenario, the SL method is undefined: it is impossible to use regression estimation to extract factor scores for η_1 and Bartlett estimation to extract factor scores for η_2 because the factor scores for this connected measurement model must be extracted simultaneously. Similarly, when there are unique factor covariances across the predictor and outcome blocks, the SL method will no longer produce unbiased estimates of the structural regression coefficients (for a formal derivation, see Appendix A).

Croon's (2002) bias-correcting approach

Rather than trying to avoid bias due to factor score indeterminacy, Croon (2002) proposed to first extract factor scores for all latent constructs using either Bartlett or regression estimation, and then correct the systematic bias in the estimated variance-covariance matrix of the factor scores analytically. Once obtained, the bias-corrected variance-covariance matrix of the predictors and outcome(s) can be used as input in either a subsequent regression analysis (see Croon, 2002; Devlieger et al., 2016) or path analysis (see Devlieger & Rosseel, 2017), yielding consistent estimates of all model parameters.

Unlike Skrondal and Laake, Croon's original work focused exclusively on deriving bias-correction formulas in factor models exhibiting simple structure (for bias correction formulas in this standard case, see Croon, 2002; Devlieger et al., 2016; Devlieger & Rosseel, 2017). Croon's bias-correcting approach is just as easily applied to connected measurement models, however, as shown in a recent paper by Hayes and Usami (2020). These authors used Croon's approach to derive bias-correction formulas for the variance-covariance matrix of the latent variables in a connected measurement model. Additionally, these authors derived bias correction formulas for the covariances between factor scores extracted from separate measurement models (for formulas, see Hayes & Usami, 2020). The derivations and simulations in Hayes and Usami's paper focused exclusively on measurement models connected by correlated uniquenesses but these same formulas may be equally well applied to connected measurement models featuring nonzero cross-loadings.

For example, examine the three-factor measurement model depicted in Figure 2. As an initial case, imagine that all unique factors are conditionally independent (the two-headed arrows depicted with dotted lines at the bottom of the figure are all zero in the population). In such a case, Hayes and Usami's (2020) formulas can be applied in either of two equivalent ways. One option would be to estimate the full measurement model of the

[2]However, as shown by Skrondal and Laake (2001) the consistency of the factorwise procedure breaks down whenever the exogenous factors (estimated with the regression method) exhibit nonzero correlations with each other, making the blockwise method a more generally applicable approach.

three factors simultaneously and correct the entire variance-covariance matrix with a single matrix formula (see Equation (8) in Hayes & Usami, 2020). A second option would be to first extract factor scores from the unconnected factor model for η_W and then from the connected measurement model for η_1 and η_2 in two separate steps, correcting the (co)variances within each model and across models using the formulas found in Hayes and Usami (2020) Appendix B. This latter option is akin to a bias-correcting (rather than bias-avoiding) version the blockwise SL approach, with the advantage that the connected measurement blocks can occur anywhere in a larger structural model – among subsets of the predictors, subsets of the outcomes, or even connected measurement models including both predictor and outcome variables. As such, this approach allows the incorporation of connected measurement structures while retaining the appeal of the model-by-model extraction approach that characterizes the classic Croon FSR method.

Hoshino and Bentler's (2013) method

In the case of continuous indicators, Hoshino and Bentler (2013) FSR method may be construed as a hybrid of the bias-avoiding and bias-correcting approaches. These authors noted that, under simple structure of the measurement indicators and conditional independence of the unique factors, the off-diagonal elements of the covariance matrix of Bartlett factor scores will be unbiased but the variances of the Bartlett factor scores will not be. Rather than correcting all entries of the Bartlett covariance matrix using analytic formulas, Hoshino and Bentler proposed substituting the estimated factor variances from the initial runs of the factor (or measurement) models at step 1 of the FSR on the diagonal elements of the Bartlett matrix. In this way, the Hoshino-Bentler (hereafter *HB*) method corrects bias in the diagonal elements of the factor score covariance matrix while avoiding bias in the off-diagonal elements through the use of Bartlett estimation.

The HB method is predicated upon the assumption that the covariances of the Bartlett factor scores will be unbiased, but this assumption does not always prove true. Hayes and Usami (2020) demonstrated that the HB method is biased whenever there are nonzero unique factor covariances across factor models (for example, when a unique factor loading on an indicator of η_W covaries with a unique factor loading on an indicator of η_1 or η_2, as depicted by the two-headed arrows at the bottom of Figure 2).

Similarly, it turns out that the HB method will be predictably biased when measurement models are connected by cross-loadings. As we show in Appendix B, in the presence of cross-loadings the covariances of the Bartlett factor scores are no longer unbiased. Because the HB method only corrects the variances on the diagonal of the Bartlett covariance matrix, this method will not completely ameliorate the bias induced by cross-loadings, even if the measurement model from which the Bartlett factor scores are extracted is correctly specified to include all truly nonzero cross loadings.

Summary of FSR approaches

To summarize, the blockwise SL method effectively avoids bias in FSR models that contain connected measurement structures within the IV and DV blocks. When these blocks are connected by cross-loadings or unique factor covariances, however, the method breaks down. The HB method will be biased in measurement models connected by cross-loadings or correlated uniquenesses. Finally, the Croon (2002) approach, implemented with Hayes and Usami's (2020) extension to connected measurement structures, should correct for bias in models containing cross-loadings, unique factor covariances, or both.

Demonstration

In this section, we will demonstrate the principles outlined above by applying SEM and FSR methods to population matrices simulated based on the models of Figures 1 and 2. We chose this population matrix approach over a more extensive simulation because our primary focus was on demonstrating the relative bias of these estimation methods, rather than on the effects of repeated sampling, per se (but see Hayes & Usami, 2020 for Monte Carlo simulation results in connected measurement models with correlated uniquenesses). Using this approach, we will see that some of these estimators remain biased even when calculations are performed on the true population matrices (rather than on imperfect sample estimates) under correct specification. Although we assess our SEM and FSR models using the configurations in Model A and Model B of Figure 1a,b, we generated our population examples under the fully latent measurement models depicted in Figure 2. We generated all models using R statistical software (R Core Team, 2013) and provide R code for all population examples in the online supplemental material.

Data generation for the demonstration

For all examples, we began by specifying the covariance matrix of the latent factors, η_1, η_2, and η_W, depicted in Figure 2. We specified a moderate correlation of .3 between η_1 and η_2 (Cohen, 1988). We set the covariances between both η_1 and η_2 and the third variable, η_W, equal to $\sqrt{.3}$, such that the entire covariance between η_1 and η_2 would be explained by W in model A, leaving zero disturbance covariance, and such that both standardized partial regression coefficients in Model B would equal .42, representing moderate effects. All three variables were therefore standardized and distributed as:

$$\begin{bmatrix} W \\ \eta_1 \\ \eta_2 \end{bmatrix} \sim MVN\left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & \sqrt{.3} & \sqrt{.3} \\ \sqrt{.3} & 1 & .3 \\ \sqrt{.3} & .3 & 1 \end{bmatrix} \right).$$

We then generated the factor loading matrix. Let λ be the factor loading for a given item that does not cross-load in the model. For indicators that do cross-load in the model, let λ_p be the loading on the primary factor and let λ_c represent the cross-loading on the second factor. Across the conditions we used to generate our population matrices, we set all $\lambda = .61$, all $\lambda_p = .49$, and all $\lambda_c = .24$ – a value almost half as large as the primary loading. The rationale for these values was that, with four standardized indicators per latent factor, tau-equivalent loadings of .61 would result in a value of

Table 1. Bias in structural parameters in simultaneous SEM models assuming simple structure of loadings and conditional independence of uniquenesses.

		Model A					Model B			
		First Factor Cross-Loads		Both Factors Cross-Load			First Factor Cross-Loads		Both Factors Cross-Load	
	Par	Est	PB	Est	PB	Par	Est	PB	Est	PB
	Conditionally Independent Uniquenesses									
γ_1	0.55	0.62	**14.08**	0.63	**14.16**	0.42	0.47	**10.97**	0.37	**−12.13**
γ_2	0.55	0.55	0.00	0.63	**14.16**	0.42	0.31	**−25.90**	0.37	**−12.13**
	Correlated Uniquenesses									
γ_1	0.55	0.73	**33.45**	0.73	**32.73**	0.42	0.54	**27.37**	0.44	3.64
γ_2	0.55	0.66	**20.80**	0.73	**32.73**	0.42	0.40	−4.42	0.44	3.64

Note: Par = population parameter, Est = estimate, PB = percent bias. Bold entries indicate absolute values of percent bias > 10, considered problematic.

Cronbach's (1951) $\alpha = .7$ under simple structure, representing a commonly endorsed lower bound for acceptable reliability. Furthermore, for indicators that cross-loaded, the values of .49 and .24 resulted in the same percentage variance explained in the item as did $\lambda = .61$ for the non-cross-loading indicators. That is, the percent variance explained in each indicator in every population model was approximately $(.61)^2 \times 100 = 37.21\%$. As such, all unique factor variances were set at approximately $1 - (.61)^2 \approx .63$, regardless of whether a given indicator featured a cross-loading. Using these values resulted in a manifest variable covariance matrix with standardized indicators (1s on the diagonals).

On the basis of these values, we generated four sets of population matrices by crossing two simulation factors. First, we varied the configuration of cross-loadings. In an initial set of conditions, we generated population matrices based on a model in which the solid gray loadings in Figure 2 were nonzero but the dashed gray loadings were equal to zero in the population. In a second set of conditions, we generated population matrices based on a model in which both the solid and the dashed gray lines in Figure 1a,b were nonzero in the population.

Second, we also varied the presence versus absence of unique factor covariances in Figure 2. In a first set of conditions, all unique factor covariances were set to zero, representing conditional independence. In a second set of conditions, we set all unique factor covariances in Figure 2 (depicted by two-headed arrows with dotted lines) to a value of .32. Given the unique factor variances of .63, unique covariances of .32 are equivalent to unique factor correlations of approximately $r \approx \frac{.3}{\sqrt{.6}\sqrt{.6}} = .5$, a strong correlation. Thus, we generated our population matrices on the basis of a fully crossed 2 (indicators of only η_1 cross-load vs. indicators of both factors cross-load) $\times$ 2 (unique factor covariances absent vs. present) factorial design with 4 cells.

Demonstrative analyses and outcomes

We fit both Model A and Model B to each simulated data matrix using two different estimation approaches with the lavaan package (Rosseel, 2012).[3] First, we fit these models using simultaneous SEM estimation assuming simple structure and conditional independence of the unique factors. Second, we fit these models using each of the three FSR methods, conducting all computations using the true population matrices (including all cross-loadings and unique factor covariances). Thus, the first set of analyses assessed bias in misspecified SEMs that omit cross-loadings and correlated uniquenesses whereas the second set of analyses examined bias in correctly specified FSR models, with all computations performed on the true measurement model matrices (see supplemental R code for exact computations).

Our main outcomes in the simulated demonstration were (1) the estimates of the structural regression parameters and (2) the percent bias of the structural regression parameter estimates, defined using the equation:

$$\text{Percent Bias}\left(\gamma_j\right) = \frac{\hat{\gamma}_j - \gamma_j}{\gamma_j} \times 100 \tag{3}$$

where $\hat{\gamma}_j$ is the estimate of either γ_1 or γ_2 returned by a given estimator fit to a given simulated population matrix and where γ_j is the true population value. Absolute values of percent bias greater than 10 are considered problematic (Muthén, Kaplan, & Hollis, 1987). In addition to percent bias of the structural parameters, we also assessed the estimated values of the exogenous (Model B) and disturbance (Model A) covariance between η_1 and η_2 estimated under-misspecified SEM and correctly specified HB FSR.

Results of the demonstration

Table 1 presents the percent bias of the structural regression parameters in each simulated condition for Models A and B, respectively, estimated using misspecified SEM models imposing simple structure and conditionally independent uniquenesses. First, examine the top half of Table 1, corresponding to population structures with cross-loadings but no unique factor covariances. In these conditions, as expected, when simple structure was imposed in Model A, the structural regression coefficients exhibited problematic positive bias for any factor whose indicators featured omitted cross-loadings. When simple structure was imposed in Model B, absolute levels of bias were again problematic, with the direction of the bias depending upon whether only one factor model or both factor models featured indicators with omitted cross-loadings. When only the indicators of η_1 featured omitted cross-loadings, estimates of γ_1 were positively biased but estimates of γ_2 were negatively biased. By contrast, when both factor models included indicators with omitted cross-loadings, both structural regression coefficients exhibited

[3]We fit all models using covariance matrix input, arbitrarily setting sample.nobs = 500. This arbitrary hypothetical *N* would, of course, have no impact on the bias of the estimates, our primary outcome in the demonstration.

negative bias in the simple structure models. Examining the bottom half of Figure 1, we see that when (positive) nonzero unique covariances were added to the model, all estimates were inflated in the positive direction. This generally increased the absolute magnitude of the positively biased coefficients and counteracted the negative bias in coefficients that were previously attenuated in models with cross-loadings but no unique covariances.

Table 2 summarizes percent bias by condition for SL, HB, and Croon methods calculated using the true population matrices (i.e., correct model specification and perfect estimation). First, examine the top half of Table 2, corresponding to population models with nonzero cross-loadings but conditionally independent unique factors. Examining the results for models A and B under these conditions, we see that the blockwise SL method produces unbiased estimates of the structural coefficients in both models, because all cross-loadings occur within-block (i.e., in the DV block for Model A and the IV block for Model B).

The HB method produces unbiased estimates of the structural coefficients in Model A but biased estimates in Model B. The reason for this is that, under these conditions, the Bartlett covariances between η_W and each connected measurement factor, η_1 and η_2, remain unbiased whereas the Bartlett covariance between η_1 and η_2 becomes biased as a result of the cross-loadings connecting η_1 and η_2. As such, the bivariate latent regressions of η_1 and η_2 on η_W in Model A produce unbiased coefficients whereas the regression of η_W on η_1 and η_2 produces biased coefficients as a result of partialling out overlapping variance from η_1 and η_2 that is estimated incorrectly. Turning to the bottom half of Table 2, we see that the SL and HB methods exhibit problematic positive bias when there are nonzero across-block unique factor covariances. By contrast, the Croon method, applied using Hayes and Usami (2020) connected measurement formulas, produces unbiased estimates in every cell of Table 2.

Finally, Table 3 presents estimates of the latent variable disturbance covariance between η_1 and η_2 in Model A and estimates of the exogenous covariance between η_1 and η_2 from Model B from (a) SEM models that incorrectly imposed simple structure in Model A, and (b) the HB method, calculated using the true (correctly specified) population matrices. For the Model A disturbance covariance (row 1), the estimates from misspecified SEM models were consistently positively biased whereas the estimates from the HB method were consistently negatively biased across conditions. In all cases, these methods suggested nonzero values for the disturbance covariance despite that the true population value was 0. A similar pattern can be found for the exogenous covariance estimated in Model B: misspecified SEM models overestimated the true factor correlation whereas correctly specified HB models underestimated this covariance across conditions.

General discussion

The theoretical discussion and simulated demonstrations presented in our paper clearly show that structural regression coefficients from SEM models that impose simple structure and omit important non-zero cross-loadings will generally be biased, with the strength and direction of this bias varying as a function of the configuration of omitted cross-loadings in the model. Our theoretical discussion and simulated demonstrations also provided a clear demonstration of the conditions under which the SL and HB methods will exhibit bias, even when calculated under correct specification using the true population values. By contrast, Croon FSR estimation implemented using Hayes and Usami's (2020) connected measurement formulas remained unbiased in all conditions. Although the SL and HB methods are well known in the literature, to our knowledge there are no other existing demonstrations of their relative shortcomings under connected measurement models. Furthermore, we are unaware

Table 2. Percent bias in structural parameters in SL, HB, and Croon FSR models under correct measurement model specification.

	Model A						Model B					
	First Factor Cross-Loads			Both Factors Cross-Load			First Factor Cross-Loads			Both Factors Cross-Load		
	SL	HB	Croon	SL	HB	Croon	SL	HB	Croon	SL	HB	Croon
	Conditionally Independent Uniquenesses											
γ_1	0	0	0	0	0	0	0	6.57	0	0	**17.57**	0
γ_2	0	0	0	0	0	0	0	6.57	0	0	**17.57**	0
	Correlated Uniquenesses											
γ_1	**20.84**	**20.26**	0	**15.96**	**15.96**	0	**23.35**	**25.09**	0	**15.96**	**26.88**	0
γ_2	**14.99**	**15.45**	0	**15.96**	**15.96**	0	**12.48**	**16.54**	0	**15.96**	**26.88**	0

Note: SL = Skrondal–Laake method, HB = Hoshino–Bentler method, Croon = Croon method with Hayes & Usami correction. Bold entries indicate absolute values of percent bias > 10, considered problematic.

Table 3. Disturbance and exogenous covariances between η_1 and η_2 in Model A and Model B by condition and estimation method.

		Conditionally Independent Uniquenesses				Correlated Uniquenesses			
		First Factor Cross-Loads		Both Factors Cross-Load		First Factor Cross-Loads		Both Factors Cross-Load	
	Par	SEM	HB	SEM	HB	SEM	HB	SEM	HB
Model A: $\sigma_{d_{\eta_1} d_{\eta_2}}$	0	0.25	−0.11	0.49	−0.28	0.00	−0.25	0.29	−0.36
Model B: $\sigma_{\eta_1 \eta_2}$	0.30	0.50	0.22	0.69	0.11	0.48	0.27	0.66	0.19

Note: Par = population parameter, $\sigma_{d_{\eta_1} d_{\eta_2}}$ = disturbance covariance between η_1 and η_2 in Model A, $\sigma_{\eta_1 \eta_2}$ = exogenous covariance between η_1 and η_2 in Model A, SEM = misspecified SEM estimation assuming simple structure and conditional independence, HB = Hoshino–Bentler FSR, with all constituent measurement models correctly specified.

of any examples in the literature of Croon FSR applied to connected measurement models featuring cross-loadings. Given the limitations of SL and HB estimation demonstrated above and the increasing popularity of Croon's (2002) method as a viable general approach to FSR estimation, we believe these findings represent an important clarification and extension of previous work in this area.

References

Asparouhov, T., & Muthén, B. (2009). Exploratory structural equation modeling. *Structural Equation Modeling*, *16*, 397–438. doi:10.1080/10705510903008204

Asparouhov, T., Muthén, B., & Morin, A. J. S. (2015). Bayesian structural equation modeling with cross-loadings and residual covariances: Comments on Stromeyer et al. *Journal of Management*, *41*, 1561–1577. doi:10.1177/0149206315591075

Bartlett, M. S. (1937). The statistical conception of mental factors. *British Journal of Psychology*, *28*, 97–104. doi:10.1111/j.2044-8295.1937.tb00863.x

Cohen, J. (1988). *Statistical power analysis for the behavioral sciences* (2nd ed.). Hillsdale, NJ: Lawrence Erlbaum Associates, Inc.

Cronbach, L. J. (1951). Coefficient alpha and the internal structure of tests. *Psychometrika*, *16*, 297–334. doi:10.1007/BF02310555

Croon, M. (2002). Using predicted latent scores in general latent structure models. In G. Marcoulides & I. Moustaki (Eds.), *Latent variable and latent structure modeling* (pp. 195–223). Mahwah, NJ: Erlbaum.

Devlieger, I., Mayer, A., & Rosseel, Y. (2016). Hypothesis testing using factor score regression: A comparison of four methods. *Educational and Psychological Measurement*, *76*, 741–770. doi:10.1177/0013164415607618

Devlieger, I., & Rosseel, Y. (2017). Factor score path analysis: An alternative for SEM? *Methodology*, *13*, 31–38. doi:10.1027/1614-2241/a000130

Devlieger, I., Talloen, W., & Rosseel, Y. (2019). New developments in factor score regression: Fit indices and a model comparison test. *Educational and Psychological Measurement*, *79*, 1017–1037. Advanced online publication doi: 10.1177/0013164419844552

Hayes, T., & Usami, S. (2020). Factor score regression in the presence of correlated unique factors. *Educational and Psychological Measurement*, *80*, 5–40. doi:10.1177/0013164419854492

Hoshino, T., & Bentler, P. M. (2013). Bias in factor score regression and a simple solution. In A. R. de Leon & K. C. Chough (Eds.), *Analysis of mixed data: Methods & applications* (pp. 43–61). Boca Raton, FL: Chapman & Hall. doi:10.1201/b14571-5

Jacobucci, R., Grimm, K. J., & McArdle, J. J. (2016). Regularized structural equation modeling. *Structural Equation Modeling*, *23*, 555–566. doi:10.1080/10705511.2016.1154793

Kenny, D. A. (1979). *Correlation and causality*. New York, NY: Wiley.

Lu, I. R. R., Kwan, E., Thomas, D. R., & Cedzynski, M. (2011). Two new methods for estimating structural equation models: An illustration and a comparison with two established methods. *International Journal of Research in Marketing*, *28*, 258–268. doi:10.1016/j.ijresmar.2011.03.006

Marsh, H. W., Morin, A. J. S., Parker, P., & Kaur, G. (2014). Exploratory structural equation modeling: An integration of the best features of exploratory and confirmatory factor analysis. *Annual Review of Clinical Psychology*, *10*, 85–110. doi:10.1146/annurev-clinpsy-032813-153700

Morin, A. J. S., Marsh, H. W., & Nagengast, B. (2013). Exploratory structural equation modeling. In G. R. Hancock & R. O. Mueller (Eds.), *Structural equation modeling: A second course* (2nd ed., pp. 395–436). Charlotte, NC: Information Age Publishing.

Mulaik, S. A. (Ed.). (2009). *Foundations of factor analysis* (2nded.). Boca Raton, FL: CRC Press.

Muthén, B., & Asparouhov, T. (2012). Bayesian structural equation modeling: A more flexible representation of substantive theory. *Psychological Methods*, *17*, 313–335. doi:10.1037/a0026802

Muthén, B., Kaplan, D., & Hollis, M. (1987). On structural equation modeling with data that are not missing completely at random. *Psychometrika*, *52*, 431–462. doi:10.1007/BF02294365

R Core Team. 2013. *R: A language and environment for statistical computing*. Vienna, Austria: R Foundation for Statistical Computing. Retrieved from http://r-project.org/

Rosseel, Y. (2012). Lavaan: An R package for structural equation modeling. *Journal of Statistical Software*, *48*, 1–36. doi:10.18637/jss.v048.i02

Scharf, F., & Nestler, S. (2019). Should regularization replace simple structure rotation in exploratory factor analysis? *Structural Equation Modeling*, 1–15. doi:10.1080/10705511.2018.1558060

Skrondal, A., & Laake, P. (2001). Regression among factor scores. *Psychometrika*, *66*, 563–575. doi:10.1007/BF02296196

Steiger, J. H., & Schönemann, P. H. (1978). A history of factor indeterminacy. In S. Shye (Ed.), *Theory construction and data analysis in the behavioural sciences* (pp. 136–178). San Francisco, CA: Jossey-Bass Publishers.

Thurstone, L. L. (1935). *The vectors of mind: Multiple-factor analysis for the isolation of primary traits*. Chicago: University of Chicago Press.

Thurstone, L. L. (1947). *Multiple-factor analysis*. Chicago, IL: University of Chicago Press.

Wright, S. (1934). The method of path coefficients. *The Annals of Mathematical Statistics*, *5*, 161–215. Retrieved from http://www.jstor.org/stable/2957502

Appendix A. Bias in the SL method in the case of across-block correlated uniquenesses

In this appendix, we show that the blockwise SL method will be biased in the presence of across-block correlated uniquenesses. Let $\mathbf{F}_\xi^R$ represent the matrix of factor scores for all latent independent variables (IVs) in the structural regression model, estimated using the regression estimator, let $\mathbf{F}_\eta^B$ represent the matrix of factor scores for all latent dependent variables (DVs) in the structural regression model, estimated using the Bartlett estimator, let $\mathbf{A}_\xi^R$ represent the factor scoring matrix for the IVs in the model and let $\mathbf{A}_\eta^B$ represent the factor scoring matrix for the DVs in the model. The expected covariance between $\mathbf{F}_\xi$ and $\mathbf{F}_\eta$ equals:

$$\mathrm{cov}\left(\mathbf{F}_\xi^R,\mathbf{F}_\eta^B\right)=\mathrm{cov}\left(\mathbf{A}_\xi^R\mathbf{x},\mathbf{A}_\eta^B\mathbf{y}\right)=\mathbf{A}_\xi^R\mathrm{cov}(\mathbf{x},\mathbf{y})\mathbf{A}_\eta^{B'}=\mathbf{A}_\xi^R\left(\boldsymbol{\Lambda}_x\Phi_{\xi\eta}\boldsymbol{\Lambda}_y'+\Theta_{xy}\right)\mathbf{A}_\eta^{B'}=\mathbf{A}_\xi^R\boldsymbol{\Lambda}_x\Phi_{\xi\eta}\boldsymbol{\Lambda}_y'\mathbf{A}_\eta^{B'}+\mathbf{A}_\xi^R\boldsymbol{\Theta}_{xy}\mathbf{A}_\eta^{B'} \tag{A1}$$

where the final two identities follow from the standard definition of the common factor model, in which $\boldsymbol{\Lambda}_x$ and $\boldsymbol{\Lambda}_y$ are the factor loading matrices for the $\mathbf{x}$ and $\mathbf{y}$ measurement models (i.e., the IV and DV blocks), respectively, $\Phi_{\xi\eta}$ is the across-block covariance matrix, and Θ_{xy} is the matrix of unique factor covariances across blocks. When there are no across-block unique factor covariances, $\boldsymbol{\Theta}_{xy}=0$, and the second term in (A1) disappears, leaving:

$$\mathrm{cov}\left(\mathbf{F}_\xi^R,\mathbf{F}_\eta^B\right)=\mathbf{A}_\xi^R\boldsymbol{\Lambda}_x\Phi_{\xi\eta}\boldsymbol{\Lambda}_y'\mathbf{A}_\eta^{B'} \tag{A2}$$

Skrondal and Laake (2001, p. 569) demonstrated that, in this case, post-multiplying $\mathrm{cov}\left(\mathbf{F}_\xi^R,\mathbf{F}_\eta^B\right)$ by the inverse of the variance of the regression-estimated IVs yields the matrix of structural regression parameters, $\boldsymbol{\Gamma}$. That is:

$$\mathrm{cov}\left(\mathbf{F}_\xi^R,\mathbf{F}_\eta^B\right)\mathrm{var}\left(\mathbf{F}_\xi^R\right)^{-1}=\mathbf{A}_\xi^R\boldsymbol{\Lambda}_x\Phi_{\xi\eta}\boldsymbol{\Lambda}_y'\mathbf{A}_\eta^{B'}\mathrm{var}\left(\mathbf{F}_\xi^R\right)^{-1}=\boldsymbol{\Gamma} \tag{A3}$$

It is easy to see that when there are some across-block unique factor covariances, $\boldsymbol{\Theta}_{xy}\neq 0$ and the equation for the structural regression parameters becomes:

$$\mathrm{cov}\left(\mathbf{F}_\xi^R,\mathbf{F}_\eta^B\right)\mathrm{var}\left(\mathbf{F}_\xi^R\right)^{-1}=\mathbf{A}_\xi^R\boldsymbol{\Lambda}_x\Phi_{\xi\eta}\boldsymbol{\Lambda}_y'\mathbf{A}_\eta^{B'}\mathrm{var}\left(\mathbf{F}_\xi^R\right)^{-1}+\mathbf{A}_\xi^R\boldsymbol{\Theta}_{xy}\mathbf{A}_\eta^{B'}\mathrm{var}\left(\mathbf{F}_\xi^R\right)^{-1}=\boldsymbol{\Gamma}+\mathbf{A}_\xi^R\boldsymbol{\Theta}_{xy}\mathbf{A}_\eta^{B'}\mathrm{var}\left(\mathbf{F}_\xi^R\right)^{-1}. \tag{A4}$$

Equation (A4) clearly demonstrates that the SL estimate of the structural parameters will be biased in the presence of across-block unique factor covariances. It is only when $\boldsymbol{\Theta}_{xy}=0$ under conditional independence of the across-block uniquenesses that the second term in Equation (A4) vanishes and the SL estimate reduces to the true structural parameter matrix, $\boldsymbol{\Gamma}$. As mentioned in the body of the paper, the SL method is undefined when there are nonzero cross-loadings across the IV and DV blocks. In such a case, the factor scores must be extracted from the entire across-block connected measurement model and there is no possibility of estimating the IV block with the regression method and the DV block with the Bartlett method.

Appendix B. Bias in the HB method in the presence of cross-loadings

The HB method for conducting FSR involves extracting all factor scores using the Bartlett estimator, and then applying a correction to the variances on the diagonal of the covariance matrix of the factor scores. This method produces unbiased estimates of the structural parameters when the measurement models are not connected by correlated uniquenesses or cross-loadings (Hoshino & Bentler, 2013). Hayes and Usami (2020) demonstrated that the HB method is biased in the presence of nonzero unique factor covariances across factor models. In this Appendix, we show analytically that the HB method will also produce biased estimates in connected measurement models featuring cross-loadings. Hayes and Usami (2020, Equations (B7) and (B9)) showed that the expected variance-covariance matrix of a connected measurement model estimated using the Bartlett method can be written:

$$\mathrm{cov}\left(\mathbf{F}_\eta^B,\mathbf{F}_\eta^B\right)=\boldsymbol{\Phi}+\mathbf{A}_\eta^B\boldsymbol{\Theta}\mathbf{A}_\eta^{B'} \tag{B1}$$

where $\mathbf{F}_\eta^B$ are, once again, the Bartlett-estimated factor scores from the connected measurement model, $\boldsymbol{\Phi}$ is the variance-covariance matrix of the latent variables in the measurement model, that is, $\mathrm{cov}(\eta,\eta)$, $\mathbf{A}_\eta^B$ is, once again, the Bartlett factor scoring matrix, and $\boldsymbol{\Theta}$ is the variance-covariance matrix of the unique factors.

As shown by Hayes and Usami (2020, footnote 7), when a model contains no cross-loadings or unique factor covariances, the second term, $\mathbf{A}_\eta^B\boldsymbol{\Theta}\mathbf{A}_\eta^{B'}$, results in a matrix with zeroes in the off-diagonals and non-zero elements in the diagonals. For example, with two latent factors, η_1 and η_2, in a given measurement model, this term may be written as the partitioned matrix product:

$$\mathbf{A}_\eta^B\boldsymbol{\Theta}\mathbf{A}_\eta^{B'}=\begin{bmatrix}\mathbf{A}_{\eta_1}^B & 0\\ 0 & \mathbf{A}_{\eta_2}^B\end{bmatrix}\begin{bmatrix}\boldsymbol{\Theta}_{\eta_1} & 0\\ 0 & \boldsymbol{\Theta}_{\eta_2}\end{bmatrix}\begin{bmatrix}\mathbf{A}_{\eta_1}^{B\,\prime} & 0\\ 0 & \mathbf{A}_{\eta_2}^{B\,\prime}\end{bmatrix}=\begin{bmatrix}\mathbf{A}_{\eta_1}^B\boldsymbol{\Theta}_{\eta_1}\mathbf{A}_{\eta_1}^{B\,\prime} & 0\\ 0 & \mathbf{A}_{\eta_2}^B\boldsymbol{\Theta}_{\eta_2}\mathbf{A}_{\eta_2}^{B\,\prime}\end{bmatrix} \tag{B2}$$

where $\mathbf{A}_{\eta_1}^B$ and $\mathbf{A}_{\eta_2}^B$ are row vectors of factor scoring coefficients for η_1 and η_2, respectively, and where $\boldsymbol{\Theta}_{\eta_1}$ and $\boldsymbol{\Theta}_{\eta_2}$ are the unique factor covariance matrices for η_1 and η_2, which may be assumed diagonal for our current purposes. Under these ideal circumstances, the HB correction to the diagonal elements of $\mathrm{cov}\left(\mathbf{F}_\eta^B,\mathbf{F}_\eta^B\right)$ will, in essence, erase or 'zero-out' the influence of the $\mathbf{A}_\eta^B\boldsymbol{\Theta}\mathbf{A}_\eta^{B'}$ term in Equation (B1), since this product only affects the variances in the main diagonal of the final matrix.

Hayes and Usami (2020) pointed out that the product $\mathbf{A}_\eta^B\boldsymbol{\Theta}\mathbf{A}_\eta^{B'}$ would contain nonzero off-diagonal elements in the presence of nonzero unique factor covariances across measurement models. In this case, Equation (B2) would become:

$$\mathbf{A}_\eta^B\boldsymbol{\Theta}\mathbf{A}_\eta^{B'}=\begin{bmatrix}\mathbf{A}_{\eta_1}^B & 0\\ 0 & \mathbf{A}_{\eta_2}^B\end{bmatrix}\begin{bmatrix}\boldsymbol{\Theta}_{\eta_1} & \boldsymbol{\Theta}_{\eta_1\eta_2}\\ \boldsymbol{\Theta}_{\eta_2\eta_1} & \boldsymbol{\Theta}_{\eta_2}\end{bmatrix}\begin{bmatrix}\mathbf{A}_{\eta_1}^{B\,\prime} & 0\\ 0 & \mathbf{A}_{\eta_2}^{B\,\prime}\end{bmatrix}=\begin{bmatrix}\mathbf{A}_{\eta_1}^B\boldsymbol{\Theta}_{\eta_1}\mathbf{A}_{\eta_1}^{B\,\prime} & \mathbf{A}_{\eta_1}^B\boldsymbol{\Theta}_{\eta_1\eta_2}\mathbf{A}_{\eta_2}^{B\,\prime}\\ \mathbf{A}_{\eta_2}^B\boldsymbol{\Theta}_{\eta_2\eta_1}\mathbf{A}_{\eta_1}^{B\,\prime} & \mathbf{A}_{\eta_2}^B\boldsymbol{\Theta}_{\eta_2}\mathbf{A}_{\eta_2}^{B\,\prime}\end{bmatrix} \tag{B3}$$

In such a case, the HB correction to the diagonal elements of $\mathrm{cov}\left(\mathbf{F}_\eta^B,\mathbf{F}_\eta^B\right)$ would not remove the bias resulting from the off-diagonal elements of $\mathbf{A}_\eta^B\boldsymbol{\Theta}\mathbf{A}_\eta^{B'}$. Here, we point out that even if all unique factors are conditionally independent, such that $\boldsymbol{\Theta}=\begin{bmatrix}\boldsymbol{\Theta}_{\eta_1} & 0\\ 0 & \boldsymbol{\Theta}_{\eta_2}\end{bmatrix}$ with diagonal $\boldsymbol{\Theta}_{\eta_1}$ and $\boldsymbol{\Theta}_{\eta_2}$, the product

$\mathbf{A}_\eta^B\boldsymbol{\Theta}\mathbf{A}_\eta^{B'}$ will have nonzero off-diagonals if the partitioned factor scoring matrix $\mathbf{A}_\eta^B$ contains non-zero off-diagonals. In such a case, Equation (B2) would become:

$$\begin{aligned}\mathbf{A}_\eta^B\boldsymbol{\Theta}\mathbf{A}_\eta^{B'} &= \begin{bmatrix}\mathbf{A}_{11}^B & \mathbf{A}_{12}^B\\ \mathbf{A}_{21}^B & \mathbf{A}_{22}^B\end{bmatrix}\begin{bmatrix}\boldsymbol{\Theta}_{\eta_1} & 0\\ 0 & \boldsymbol{\Theta}_{\eta_2}\end{bmatrix}\begin{bmatrix}\mathbf{A}_{11}^{B'} & \mathbf{A}_{21}^B\\ \mathbf{A}_{12}^B & \mathbf{A}_{22}^{B'}\end{bmatrix} = \begin{bmatrix}\mathbf{A}_{11}^B\boldsymbol{\Theta}_{\eta_1} & \mathbf{A}_{12}^B\boldsymbol{\Theta}_{\eta_2}\\ \mathbf{A}_{21}^B\boldsymbol{\Theta}_{\eta_1} & \mathbf{A}_{22}^B\boldsymbol{\Theta}_{\eta_2}\end{bmatrix}\begin{bmatrix}\mathbf{A}_{11}^{B'} & \mathbf{A}_{21}^B\\ \mathbf{A}_{12}^B & \mathbf{A}_{22}^{B'}\end{bmatrix}\\ &= \begin{bmatrix}\mathbf{A}_{11}^B\boldsymbol{\Theta}_{\eta_1}\mathbf{A}_{11}^{B'} + \mathbf{A}_{12}^B\boldsymbol{\Theta}_{\eta_2}\mathbf{A}_{12}^B & \mathbf{A}_{11}^B\boldsymbol{\Theta}_{\eta_1}\mathbf{A}_{21}^B + \mathbf{A}_{12}^B\boldsymbol{\Theta}_{\eta_2}\mathbf{A}_{22}^{B'}\\ \mathbf{A}_{21}^B\boldsymbol{\Theta}_{\eta_1}\mathbf{A}_{11}^{B'} + \mathbf{A}_{22}^B\boldsymbol{\Theta}_{\eta_2}\mathbf{A}_{12}^B & \mathbf{A}_{21}^B\boldsymbol{\Theta}_{\eta_1}\mathbf{A}_{21}^B + \mathbf{A}_{22}^B\boldsymbol{\Theta}_{\eta_2}\mathbf{A}_{22}^{B'}\end{bmatrix}\end{aligned} \tag{B4}$$

It turns out that $\mathbf{A}_\eta^B$ will exhibit such a form when either the model for η_1, the model for η_2 or both contain cross-loadings on the other factor. As is well-known (Bartlett, 1937; cf. Devlieger et al., 2016), the formula for the Bartlett factor scoring matrix is:

$$\mathbf{A}_\eta^B = \left(\boldsymbol{\Lambda}'\boldsymbol{\Theta}^{-1}\boldsymbol{\Lambda}\right)^{-1}\boldsymbol{\Lambda}'\boldsymbol{\Theta}^{-1}. \tag{B5}$$

To understand how this formula performs in connected measurement models containing cross-loadings, let $\lambda_{\eta_{1[ss]}}$ and $\lambda_{\eta_{2[ss]}}$ be column vectors of factor loadings for indicators in the η_1 and η_2 measurement models that exhibit simple structure (that is, indicators that do not cross-load on the other factor). For indicators that feature cross-loadings, let $\lambda_{\eta_{1[p]}}$ and $\lambda_{\eta_{2[p]}}$ be column vectors of factor loadings on the primary factor and let $\lambda_{\eta_{1[c]}}$ and $\lambda_{\eta_{2[c]}}$ be column vectors of cross-loadings on the non-primary factor. Then, we can define $\boldsymbol{\Lambda}$ as the partitioned matrix:

$$\boldsymbol{\Lambda} = \begin{bmatrix}\lambda_{\eta_{1[ss]}} & 0\\ \lambda_{\eta_{1[p]}} & \lambda_{\eta_{1[c]}}\\ 0 & \lambda_{\eta_{2[ss]}}\\ \lambda_{\eta_{2[c]}} & \lambda_{\eta_{2[p]}}\end{bmatrix}. \tag{B6}$$

With conditionally independent uniquenesses, we may define $\boldsymbol{\Theta}^{-1}$ as the partitioned matrix:

$$\boldsymbol{\Theta}^{-1} = \begin{bmatrix}\boldsymbol{\Theta}_{\eta_1}^{-1} & 0\\ 0 & \boldsymbol{\Theta}_{\eta_2}^{-1}\end{bmatrix} = \begin{bmatrix}\boldsymbol{\Theta}_{\eta_{1[ss]}}^{-1} & 0 & 0 & 0\\ 0 & \boldsymbol{\Theta}_{\eta_{1[cl]}}^{-1} & 0 & 0\\ 0 & 0 & \boldsymbol{\Theta}_{\eta_{2[ss]}}^{-1} & 0\\ 0 & 0 & 0 & \boldsymbol{\Theta}_{\eta_{2[cl]}}^{-1}\end{bmatrix} \tag{B7}$$

where $\boldsymbol{\Theta}_{\eta_1}^{-1}$ and $\boldsymbol{\Theta}_{\eta_2}^{-1}$ are diagonal matrices with the reciprocals of the diagonal elements of $\boldsymbol{\Theta}_{\eta_1}$ and $\boldsymbol{\Theta}_{\eta_2}$ on the diagonals, where $\boldsymbol{\Theta}_{\eta_{1[ss]}}^{-1}$ and $\boldsymbol{\Theta}_{\eta_{2[ss]}}^{-1}$ are the subsets of these matrices corresponding to the indicators of η_1 and η_2 that conform to simple structure and where $\boldsymbol{\Theta}_{\eta_{1[cl]}}^{-1}$ and $\boldsymbol{\Theta}_{\eta_{2[cl]}}^{-1}$ are the subsets of these matrices corresponding to the indicators of η_1 and η_2 that exhibit cross loadings.

With these definitions in hand, we can define:

$$\Lambda'\Theta^{-1} = \begin{bmatrix}\lambda'_{\eta_{1[ss]}}\boldsymbol{\Theta}_{\eta_{1[ss]}}^{-1} & \lambda'_{\eta_{1[p]}}\boldsymbol{\Theta}_{\eta_{1[cl]}}^{-1} & 0 & \lambda'_{\eta_{2[c]}}\boldsymbol{\Theta}_{\eta_{2[cl]}}^{-1}\\ 0 & \lambda'_{\eta_{1[c]}}\boldsymbol{\Theta}_{\eta_{1[cl]}}^{-1} & \lambda'_{\eta_{2[ss]}}\boldsymbol{\Theta}_{\eta_{2[ss]}}^{-1} & \lambda'_{\eta_{2[p]}}\boldsymbol{\Theta}_{\eta_{2[cl]}}^{-1}\end{bmatrix} \tag{B8}$$

and

$$\Lambda'\Theta^{-1}\Lambda = \begin{bmatrix}\lambda'_{\eta_{1[ss]}}\boldsymbol{\Theta}_{\eta_{1[ss]}}^{-1}\lambda_{\eta_{1[ss]}} + \lambda'_{\eta_{1[p]}}\boldsymbol{\Theta}_{\eta_{1[cl]}}^{-1}\lambda_{\eta_{1[p]}} + \lambda'_{\eta_{2[p]}}\boldsymbol{\Theta}_{\eta_{2[cl]}}^{-1}\lambda_{\eta_{2[c]}} & \lambda'_{\eta_{1[p]}}\boldsymbol{\Theta}_{\eta_{1[cl]}}^{-1}\lambda_{\eta_{1[c]}} + \lambda'_{\eta_{2[c]}}\boldsymbol{\Theta}_{\eta_{2[cl]}}^{-1}\lambda_{\eta_{2[p]}}\\ \lambda'_{\eta_{1[c]}}\boldsymbol{\Theta}_{\eta_{1[cl]}}^{-1}\lambda_{\eta_{1[p]}} + \lambda'_{\eta_{2[p]}}\boldsymbol{\Theta}_{\eta_{2[cl]}}^{-1}\lambda_{\eta_{2[c]}} & \lambda'_{\eta_{1[c]}}\boldsymbol{\Theta}_{\eta_{1[cl]}}^{-1}\lambda_{\eta_{1[c]}} + \lambda'_{\eta_{2[ss]}}\boldsymbol{\Theta}_{\eta_{2[ss]}}^{-1}\lambda_{\eta_{2[ss]}} + \lambda'_{\eta_{2[p]}}\boldsymbol{\Theta}_{\eta_{2[cl]}}^{-1}\lambda_{\eta_{2[p]}}\end{bmatrix} \tag{B9}$$

It is clear from Equation (B9) that even if one factor model contained no cross-loading indicators – e.g., say $\boldsymbol{\Lambda}_{\eta_2} = \left[\lambda_{\eta_{2[ss]}}\right]$ such that all terms involving $\lambda_{\eta_{2[p]}}$ and $\lambda_{\eta_{2[c]}}$ vanish from Equation (B9) – the matrix product $\boldsymbol{\Lambda}'\boldsymbol{\Theta}^{-1}\boldsymbol{\Lambda}$ will contain nonzero off-diagonal elements with the result that its inverse $\left(\boldsymbol{\Lambda}'\boldsymbol{\Theta}^{-1}\boldsymbol{\Lambda}\right)^{-1}$ will also contain nonzero off-diagonal elements. From these equations, we can see that formula (B5) for the Bartlett factor score matrix, $\mathbf{A}_\eta^B$, will return a result with nonzero entries in both the diagonal and off-diagonal elements of the partitioned matrix. For this reason, the term $\mathbf{A}_\eta^B\boldsymbol{\Theta}\mathbf{A}_\eta^B$ in Equation (B1) will create bias in both the diagonal and off-diagonal elements of the covariance matrix of the Bartlett factor scores. Correcting only the variances on the diagonal using the HB method will fail to completely rid Equation (1) of bias because, in such a situation, the Bartlett factor score covariances are no longer unbiased estimates of the true latent factor covariances.